Grešak, Kožar, Tiegl: ELEMENTARNE FUNKCIJE. KOMPLEKSNA ŠTEVILA

Poglavje IX: Kompleksna števila; stran 70, naloga19: Razcepi izraze v obsegu kompleksnih števil.

$a)\ a^2+36$ $b)\ x^2+100$ $c)\ 25b^2+1$

č) $81x^2+625$ $d)\ x^2+2$ $e)\ x^2+50$

$f)\ x^2+12$ $g)\ 49x^2+5y^2$ $h)\ x^4+9x^2$

$i)\ 2b^4+162b^2$ $j)\ x^4+10x^2+9$ $k)\ x^4-x^2-20$

$l)\ x^4+35x^2-36$ $m)\ x^3-8$ $n)\ x^3+27$

$o)\ x^4+x$ $p)\ x^5-64x^2$

Teorija:

Izraz x^2+1 ni razcepen v množici realnih števil. Enačba $x^2+1=0$ ni rešljiva v množici realnih števil $\mathbb{R}$.

Tako imamo razlog za vpeljavo nove številske množice kompleksnih števil $\mathbb{C}$, v kateri velja $i^2=-1$, kjer i imenujemo imaginarna enota in $i=\sqrt{-1}$.

Sedaj lahko x^2+1 zapišemo kot:

$x^2+1=x^2-(-1)=x^2-i^2=(x-i)(x+i)$

Vidimo, da je razcep izraza x^2+1 možen v množici kompleksnih števil.

Prav tako lahko z uporabo $i^2=-1$ $\left(i=\sqrt{-1}\right)$ rešimo enačbo $x^2+1=0$. Levo stran zapišemo kot $x^2-i^2=0$ in razcepimo $(x-i)(x+i)=0$ ter dobim $x-i=0$ ali $-x+i=0$ in s tem dve kompleksni rešitvi $x_1=i$ in $x_2=-i$.

Za zaključek:

Izraz a^2+b^2 je razcepen v kompleksnem

$a^2+b^2=a^2-bi^2=(a-bi)(a+bi)$

Z uporabo $i=\sqrt{-1}$ in seveda formul:

$a^2-b^2=(a-b)(a+b)$ $\qquad a^2+b^2=a^2-i^2b^2=(a-ib)(a+ib)$

$a-b=\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$ $\quad a^3+b^3=(a+b)(a^2-ab+b^2)$

rešimo našo nalogo. Pri tem se spomnimo tudi kvadratne enačbe

$ax^2+bx+c=0$ in njenih rešitev:

$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}$, kjer je $D=b^2-4ac$

Sedaj znamo izračunati tudi kvadratni koren iz negativnega števila.

Primer: $\sqrt{-25}=\sqrt{25i^2}=5i$

$a)\ a^2 + 36 = a^2 — (-36)$

$= a^2 - 36i^2$

$= \underline{(a - 6i)(a + 6i)}$

$b)\ x^2 + 100 = x^2 - (-10^2)$

$= x^2 — (-1 \,.\, 10^2)$

$= x^2 - 10^2 i^2$

$= \underline{(x - 10i)(x + 10i)}$

$c)\ 25b^2 + 1 = (5b)^2 - (-1)$

$= (5b)^2 - i^2$

$= \underline{(5b - i)(5b + i)}$

$č)\ 81x^2 + 625 = (9x)^2 + 25^2$

$= (9x)^2 - (-25)^2$

$= (9x)^2 - (25i)^2$

$= \underline{(3x - 5i)(3x + 5i)}$

$d)\ x^2 + 2 = x^2 - (-2)$

$= x^2 - 2i^2$

$= \underline{\left(x - i\sqrt{2}\right)\left(x + i\sqrt{2}\right)}$

$e)\ x^2 + 50 = x^2 - (-50)$

$= x^2 - 50i^2$

$= \left(x - \sqrt{50i^2}\right)\left(x + \sqrt{50i^2}\right)$ $\qquad \sqrt{50} = \sqrt{25.2} = 5\sqrt{2}$

$= \underline{\left(x - 5i\sqrt{2}\right)\left(x + 5i\sqrt{2}\right)}$

$f)\ x^2 + 12 = x^2 - (-12)$

$= x^2 - 12i^2$

$= \left(x - \sqrt{12i^2}\right)\left(x + \sqrt{12i^2}\right)$ $\qquad \sqrt{12} = \sqrt{3 \,.4} = 2\sqrt{3}$

$= \underline{\left(x - 2i\sqrt{3}\right)\left(x + 2i\sqrt{3}\right)}$

$g)\ 49x^2 + 5y^2 = 49x^2 — (-5y^2) =$

$= (7x)^2 - 5i^2y^2$

$= \left(7x - \sqrt{5i^2y^2}\right)\left(7x + \sqrt{5i^2y^2}\right)$

$= \underline{(7x - iy\sqrt{5})(7x + iy\sqrt{5})}$

$h)\ x^4 + 9x^2 = x^2(x^2 + 9)$

$= x^2\left(x^2 - (-9)\right)$

$= x^2(x^2 - 9i^2)$

$= \underline{x^2(x - 3i)(x + 3i)}$

$i)\ 2b^4 + 162b^2 = 2b^2(b^2 + 81)$

$= 2b^2\left(b^2 - (-81)\right)$

$= 2b^2(b^2 - 9i^2)$

$= \underline{2b^2(b - 3i)(b + 3i)}$

$j)\ x^4 + 10x^2 + 9 = (po\ Viètu) = (x^2 + 1)(x^2 + 9)$

$= (x^2 - i^2)(x^2 + 9i^2)$

$= \underline{(x - i)(x + i)(x - 3i)(x + 3i)}$

ali z novo neznanko $x^2 = t$:

$= t^2 + 10t + 9$

$= (t + 1)(t + 9)$

$= (x^2 + 1)(x^2 + 9)$

$= \underline{(x + i)(x - i)(x - 3i)(x + 3i)}$

$k)\ x^4 - x^2 - 20 = (x^2 + 4)(x^2 - 5)$ po Viètu $\left(-20 = 4\,.\,(-5)\right)$

$= \left(x^2 - (-4)\right)\left(x^2 - \left(\sqrt{5}\right)^2\right)$

$= (x^2 - 4i^2)\left(4 - \sqrt{5}\right)\left(4 + \sqrt{5}\right)$

$= \underline{(x - 2i)(x + 2i)\left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right)}$

$l)\ x^4 + 35x^2 - 36 =\ (vstavimo\ novo\ neznanko\ x^2 = t)$

$= t^2 + 35t - 36$

$= (t - 1)(t + 36)$ po Viètu $\left(-36 =\ -1.\,(+36)\right)$

$= (x^2 - 1)(x^2 + 36)$

$= (x - 1)(x + 1)(x^2 - 36i^2)$

$= \underline{(x - 1)(x + 1)(x - 6i)(x + 6i)}$

$m)\ x^3 - 8 = x^3 - 2^3$

$= (x-2)(x^2+2x+6)$

Rešimo kvadratno enačbo:	
$x^2+2x+4=0$	$D = b^2 - 4ac$
$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$	$D = 4 - 16$
$x_{1,2} = \frac{-2 \pm 2i\sqrt{3}}{2}$	$D = -12$
$x_{1,2} = \frac{2(-1 \pm i\sqrt{3})}{2}$	$\sqrt{D} = \sqrt{-12}$
$x_1 = -1 + i\sqrt{3}$	$= \sqrt{i^2 . 3.4}$
$x_2 = -1 - i\sqrt{3}$	$= 2i\sqrt{3}$
$\left(x - \left(-1 - i\sqrt{3}\right)\right)\left(x - \left(-1 + i\sqrt{3}\right)\right) = 0$	

$= \underline{(x-2)\left(x+1+i\sqrt{3}\right)\left(x+1-i\sqrt{3}\right)}$

$n)\ x^3 + 27 = x^3 + 3^3$

$= (x+3)(x^2-3x+9)$

$x^2+3x+9=0$	$D = 9 - 36$
$x_{1,2} = \frac{3 \pm 3i\sqrt{3}}{2}$	$D = -27$
	$\sqrt{D} = \sqrt{-27}$
$\left(x - \frac{3+3i\sqrt{3}}{2}\right)\left(x - \frac{3-3i\sqrt{3}}{2}\right) = 0$	$= i\sqrt{9 .3}$
	$= 3i\sqrt{3}$

$= \underline{(x+3)\left(x - \frac{3+3i\sqrt{3}}{2}\right)\left(x - \frac{3-3i\sqrt{3}}{2}\right)}$

$o)\ x^4 + x = x(x^3+1)$

$= x(x+1)(x^2-x+1)$

$x^2-x+1=0$	$D = 1 - 4$
$x_{1,2} = \frac{1 \pm i\sqrt{3}}{2}$	$D = -3$
	$\sqrt{D} = i\sqrt{3}$
$\left(x - \frac{1+i\sqrt{3}}{2}\right)\left(x - \frac{1-i\sqrt{3}}{2}\right) = 0$	

$= \underline{x(x+1)\left(x - \frac{1+i\sqrt{3}}{2}\right)\left(x - \frac{1-i\sqrt{3}}{2}\right)}$

$p)\ x^5 - 64x^2 = x^2(x^3 - 64)$

$= x^2(x^3 - 4^3)$

$= x^2(x-4)(x^2 + 4x + 16)$

$x^2 + 4x + 16 = 0$	$D = 16 - 64$
$x_{1,2} = \frac{-4 \pm 4i\sqrt{3}}{2}$	$D = -48$
$x_{1,2} = \frac{-4(1 \pm i\sqrt{3})}{2}$	$\sqrt{D} = \sqrt{-48}$
$x_{1,2} = 2(1 \pm i\sqrt{3})$	$= i\sqrt{16 .3}$
$x_{1,2} = 2 \pm 2i\sqrt{3}$	$= 4i\sqrt{3}$
$\left(x - (2 + 2i\sqrt{3})\right)\left(x - (2 - 2i\sqrt{3})\right) = 0$	

$= \underline{x^2(x-4)(x-2-2i\sqrt{3})(x-2+2i\sqrt{3})}$